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Brief note

MARANGONI CONVECTION IN A RELATIVELY HOTTER OR COOLER LIQUID LAYER WITH FREE BOUNDARIES

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In this paper, we study the onset of cellular convection in a horizontal fluid layer heated from below, with a free-slip boundary condition at the bottom when the driving mechanism is surface tension at the upper free surface, in the light of the modified analysis of Banerjee *et al.* (Jour. Math. & Phys. Sci., 1983, 17, 603). This leads to a formulation of the problem which depends upon whether the liquid layer is relatively hotter or cooler. It is found that the phenomenon of surface tension driven instability problems should not only depend upon the Marangoni number which is proportional to the maintained temperature differences across the layer but also upon another parameter that arises due to variation in the specific heat at constant volume on account of the variations in temperature. Numerical results are obtained for the problem wherein the lower free boundary is perfectly thermally conducting.

Key words: surface tension, convection, conducting, linear stability.

1. Introduction

The problem of the onset of convective instability in a thin layer of the fluid heated from below has its origin in the experimental observations of Bénard (1901). The mechanism of convective instability driven either by buoyancy arising due to expansion of a heated liquid or by surface tension variation with temperature has been the subject of a great deal of theoretical and experimental investigations since the pioneering works of Rayleigh (1916) and Pearson (1958). The mechanism associated with buoyancy is usually called the Rayleigh-Bénard instability since there appears a non-dimensional number R called the Rayleigh number and that associated with surface tension is called the Marangoni-Bénard instability since there appears a non-dimensional number M called the Marangoni number. Usually, in practice both buoyancy and surface tension are operative, therefore, Nield (1964) combined both mechanisms into a single analysis and found that as the depth of the layer decreases the surface tension mechanism becomes more dominant and when the depth of the layer is at most *0.1 cm* the buoyancy effect can safely be neglected for most liquids. The pioneering works of Pearson and Nield have subsequently been extended and refined by many researchers (see for example, Vidal and Acrivos (1966), and Takashima (1970; 1981)).

In almost all these works pertaining to the Marangoni-Bénard instability, the treatments used are based on the classical Rayleigh's theory and therefore, do not distinguish whether the layer of the liquid is hotter or cooler and predicts the same Marangoni number at the onset of instability in both the cases which is

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contrary to physical intuition. More recently, Pearson's problem of surface tension driven convection has been reformulated by Gupta and Shandil (2011b), which does significantly depend upon whether the liquid layer is hotter or cooler, using the modified analysis of Banerjee *et al.* (1983) given for the case of buoyancy driven convection.

In this paper, we study the onset of convection in a horizontal fluid layer heated from below, with free-slip boundary condition at the bottom when the driving mechanism is surface tension at the upper surface, in the light of the modified argument of Banerjee *et al*. (1983). It is found that the phenomenon of surface tension driven instability problems should not only depend upon the Marangoni number which is proportional to the maintained temperature differences across the layer but also upon another parameter that arises due to variation in the specific heat at constant volume on account of the variations in temperature. This application makes a provision in the theory so as to recognize the fact that a relatively hotter layer with its heat diffusivity apparently increased/decreased as a consequence of an actual decrease/increase (depending upon the liquid) in its specific heat at constant volume must exhibit convection of the type observed by Bénard at a higher/lower Marangoni number than a cooler layer under almost identical conditions otherwise.

2. Formulation of the problem

We consider an infinite horizontal layer of a viscous fluid of uniform thickness d at rest, whose lower surface is assumed to be free and perfectly heat conducting, and the upper one to be free and finitely conducting where surface tension gradients arise due to temperature perturbations. We choose a Cartesian coordinate system of axes with the x and y axes in the plane of the lower surface and the z axis along the vertically upward direction so that the fluid is confined between the planes at $z=0$ and $z=d$. A temperature gradient is maintained across the layer by maintaining the lower boundary at a constant temperature $T_0(>0)$ and the upper boundary at $T_1<(T_0)$. It is assumed that surface tension is given by the simple linear law $\tau=\tau_1-\sigma(T-T_1)$ where the constant τ_1 is the unperturbed value of τ at the unperturbed surface temperature $T=T_1$ and $-\sigma=(\partial\tau/\partial T)_{T/T_1}$ represents the rate of change of surface tension with temperature, evaluated at temperature T_1, and surface tension being a monotonically decreasing function of temperature, σ is positive.

Following Banerjee *et al.* (1983), we can write the modified and linearized equations governing the small perturbations in the relevant context (neglecting buoyancy) as

$$\left(\frac{\partial}{\partial t}-\nu\nabla^2 w\right)\nabla^2 w=0\,, \tag{2.1}$$

$$(1-\alpha_2 T_0)\left(\frac{\partial\theta}{\partial t}-\beta w\right)=\kappa\nabla^2\theta \tag{2.2}$$

where w and θ denote, respectively, the z-component of velocity perturbation and temperature perturbation; β is the temperature gradient which is maintained; ν is the kinematic viscosity; κ is the thermal diffusivity and they are each assumed constant. Further, the coefficient α_2 (due to variation in the specific heat at constant volume on account of variations in the temperature) is a constant that lies in the range of -10^{-4} to 10^{-4}; $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ and t denotes time.

It should be noted that equation of heat conduction differs from Pearson's corresponding equation by the multiplication by the factor $(1-\alpha_2 T_0)$ on the left hand side as compared to 1.

In seeking solutions of Eqs (2.1) and (2.2), we must satisfy certain boundary conditions, The boundary conditions at the lower free surface $z = 0$ are straightforward and given by

$$w = 0\,, \tag{2.3}$$

$$\frac{\partial^2 w}{\partial z^2} = 0\,, \tag{2.4}$$

$$\theta = 0 \tag{2.5}$$

The boundary conditions at the upper free surface $z = d$ are more complicated.

Because of the non-deflecting surface, the normal component of the velocity must vanish, that is,

$$w = 0\,. \tag{2.6}$$

The stress-balance condition satisfies the equation

$$\rho\nu\frac{\partial^2 w}{\partial z^2} = \sigma\nabla_1^2\theta \tag{2.7}$$

here ρ is the density and $\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. The boundary condition (2.7) is usually referred to as the Marangoni boundary condition. Finally, if we consider conservation of heat transport across the upper free surface, then we have

$$-k\frac{\partial\theta}{\partial z} = q\theta \tag{2.8}$$

where k is the thermal conductivity of the fluid and q is the heat transfer coefficient.

We now suppose that the perturbations w and θ are of the form

$$w(x,y,z,t) = w(z)\exp[i(a_x x + a_y y) + pt]\,,$$

$$\theta(x,y,z,t) = \theta(z)\exp[i(a_x x + a_y y) + pt]$$

where $a = \sqrt{a_x^2 + a_y^2}$ is the wave number of the disturbance and p is a time constant (which can be complex). We now introduce the non-dimensional quantities using d, ν/d, d^2/ν and $\beta d\nu/\kappa$ as the appropriate scales for length, velocity, time and temperature, respectively and putting $z_* = z/d$, $a_* = ad$, $p_* = pd^2/\nu$, $w_* = wd/\nu$, $\theta_* = \theta\kappa/\beta d\nu$, and $D_* = d(d/dz)$. We now let x, y and z stand for co-ordinates in the new units and omitting the asterisk for simplicity, Eqs (2.1)-(2.2) and boundary conditions (2.3)-(2.8) can be reduced to the following non-dimensional form

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - p\right)w = 0\,, \tag{2.9}$$

$$\left(D^2 - a^2 - \left(1 - \alpha_2 T_0\right)p\mathrm{P_r}\right)\theta = -(1 - \alpha_2 T_0)w\,, \tag{2.10}$$

$$w(0) = 0, \tag{2.11}$$

$$D^2 w(0) = 0 \tag{2.12}$$

$$\theta(0) = 0, \tag{2.13}$$

evaluated on the lower free boundary $z = 0$, and

$$w(1) = 0, \tag{2.14}$$

$$D^2 w(1) = -a^2 \mathrm{M}\theta(1), \tag{2.15}$$

$$D\theta(1) = -\mathrm{L}\theta(1), \tag{2.16}$$

evaluated on the upper free surface $z = 1$, where $\mathrm{M} = \dfrac{\sigma\beta d^2}{\rho\kappa\nu}$ is the Marangoni number, $\mathrm{P_r} = \dfrac{\nu}{\kappa}$ is the Prandtl number and $\mathrm{L} = \dfrac{qd}{k}$ is the Biot number.

Gupta and Shandil (2011a) established numerically that the 'principle of exchange of stabilities' is valid for the present problem, the marginal state is, therefore, characterized by $p = 0$ and Eqs (2.8) and (2.9) become

$$\left(D^2 - a^2\right)^2 w = 0, \tag{2.17}$$

$$\left(D^2 - a^2\right)\theta = -\left(1 - \alpha_2 T_0\right) w. \tag{2.18}$$

A solution to Eqs (2.17)–(2.18) is sought subject to boundary conditions (2.11)–(2.16). Thus we have an eigenvalue problem.

3. Solution of the problem

The solution of Eq.(2.17) subject to boundary conditions (2.11)-(2.12) and (2.14) is given by

$$w = A\left[\frac{C_a S_{az} - z S_a C_{az}}{S_a}\right] \tag{3.1}$$

where $S_a = \sinh a$, $C_a = \cosh a$, $S_{az} = \sinh az$, $C_{az} = \cosh az$ and A is the constant of integration.

The solution of Eq.(2.18), using the expression (3.1) for w and boundary conditions (2.13) and (2.16) is given by

$$\theta = -\frac{A}{\left(1 - \alpha_2 T_0\right)}\left[-\frac{z^2}{4a} S_{az} + \frac{S_a + 2aC_a}{4a^2 S_a} z C_{az} + \right. \\ \left. -\left\{\frac{(1 + \mathrm{L})\{C_a\left(2aC_a + S_a\right) - aS_a\} + a^2 S_a C_a}{4a^2 S_a\left(aC_a + \mathrm{L}S_a\right)}\right\} S_{az}\right]. \tag{3.2}$$

Now substitution from Eqs (3.1) and (3.2) into the remaining boundary condition (2.15) yields finally the neutral stability condition

$$\mathrm{M} = \frac{1}{\left(1-\alpha_2 T_0\right)}\left[\frac{8aS_a^2\left(aC_a + \mathrm{L}S_a\right)}{\left(C_a^3 + aS_a - (2a^2+1)C_a\right)}\right]. \tag{3.3}$$

4. Numerical results and discussion

Since some of the algebraic manipulations involved are rather lengthy a symbolic algebra package is used to compute the minimum values of the Marangoni number M with respect to the wave number a for fixed values of the parameters $\alpha_2 T_0$ and L, using relation (3.3). The critical Marangoni number $\mathrm{M_c}$ and the corresponding wave number a_c for various values (which cover usual laboratory conditions) of the parameter $\alpha_2 T_0$ when L = *0*, *2*, *4* and *6* are presented in Tab.1. It is found that, for a given value of L, an increase in the value of $\alpha_2 T_0$ leads to an increased value of $\mathrm{M_c}$ if α_2 is positive and a decrease in the value of $\alpha_2 T_0$ leads to a decreased value of $\mathrm{M_c}$ if α_2 is negative.

Table 1. The numerical values of $\mathrm{M_c}$ and a_c for various values of $\alpha_2 T_0$ and L.

$\alpha_2 T_0$	L = *0*		L = *2*		L = *4*		L = *6*	
	$\mathrm{M_c}$	a_c	$\mathrm{M_c}$	a_c	$\mathrm{M_c}$	a_c	$\mathrm{M_c}$	a_c
-0.9	*30.3148*	*1.7003*	*61.4274*	*2.0244*	*91.0147*	*2.1524*	*120.168*	*2.2247*
-0.7	*33.8812*	*1.7003*	*68.6542*	*2.0244*	*101.722*	*2.1524*	*134.306*	*2.2247*
-0.5	*38.3987*	*1.7003*	*77.808*	*2.0244*	*115.285*	*2.1524*	*152.213*	*2.2247*
-0.2	*47.9984*	*1.7003*	*97.2601*	*2.0244*	*144.107*	*2.1524*	*190.267*	*2.2247*
0	*57.598*	*1.7003*	*116.712*	*2.0244*	*172.928*	*2.1524*	*228.32*	*2.2247*
0.2	*71.9976*	*1.7003*	*145.89*	*2.0244*	*216.16*	*2.1524*	*285.4*	*2.2247*
0.5	*115.196*	*1.7003*	*233.424*	*2.0244*	*345.856*	*2.1524*	*456.64*	*2.2247*
0.7	*191.993*	*1.7003*	*389.04*	*2.0244*	*576.426*	*2.1524*	*761.066*	*2.2247*
0.9	*575.98*	*1.7003*	*1167.12*	*2.0244*	*1729.28*	*2.1524*	*2283.2*	*2.2247*

The value of the critical wave number a_c remains unchanged for various values of $\alpha_2 T_0$. The values of $\mathrm{M_c}$ and the corresponding wave number a_c when $\alpha_2 T_0 = 0$ agree with the values obtained by Boeck and Thess (1997) for the corresponding given values of L. In particular, for L = *0* we have $M_c = 57.598$ and $a_c = 1.7003$ to be compared with $\mathrm{M_c} = 79.607$ and $a_c = 1.993$ as obtained by Pearson (1958) for the conducting case of the lower rigid boundary. The higher value of the critical Marangoni number in the conducting case of lower rigid boundary is due to the stabilizing effect of friction at the bottom, which is also known from buoyancy driven Rayleigh-Bénard convection (Chandrashekhar, 1961).

As L increases, the critical Marangoni number M_c as well as the corresponding wave number a_c increase. In the limit $L \to \infty$ we find that M_c becomes asymptotically proportional to L while $a_c \to a_{max} \approx 2.5054$ remains finite. The value of M_c as a function of $\alpha_2 T_0$ in this limit ($L \to \infty$) can easily be obtained by inserting a_{max} and the value of $\alpha_2 T_0$ into Eq.(3.3).

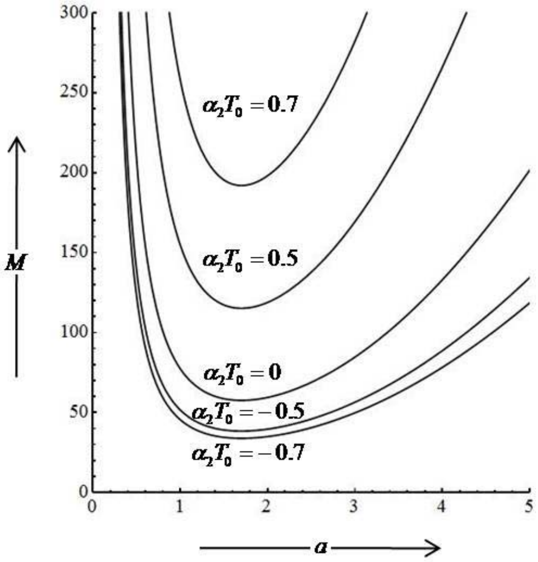

Fig.1a. Neutral stability cuwes for various values of $\alpha_2 T_0$ when L=*0*.

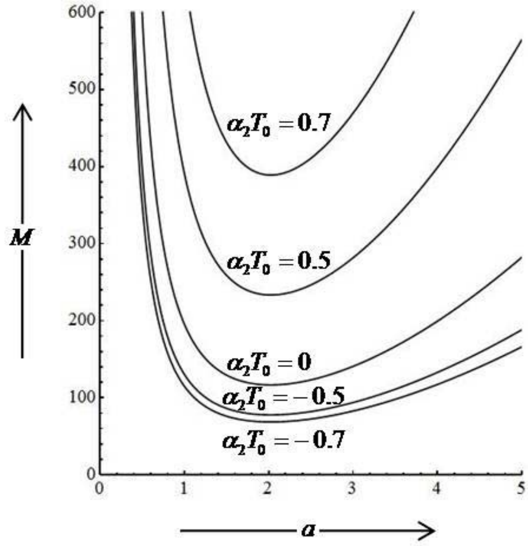

Fig.1b. Neutral stability cuwes for various values of $\alpha_2 T_0$ when L=*2*.

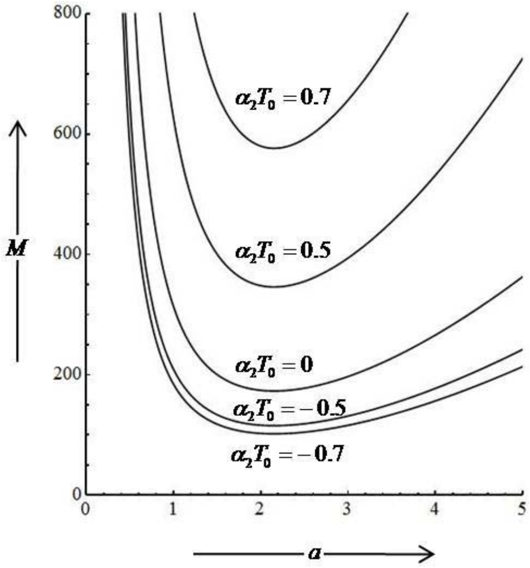

Fig.1c. Neutral stability cuwes for various values of $\alpha_2 T_0$ when L=*4*.

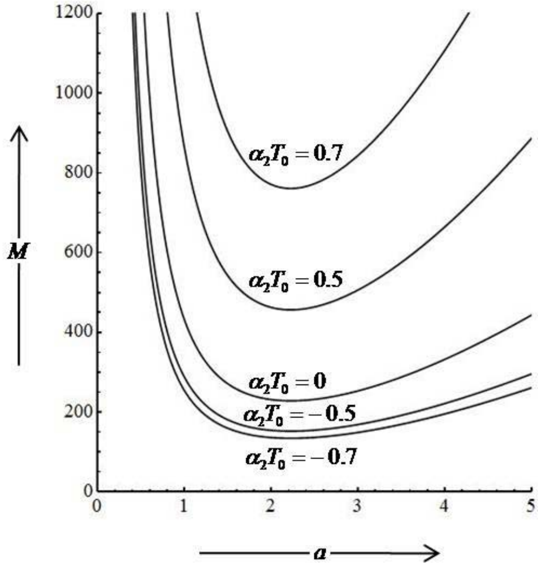

Fig.1b. Neutral stability cuwes for various values of $\alpha_2 T_0$ when L=*6*.

The relation (3.3) is plotted in Figs 1a-d, as the neutral stability curves for L = *0*, *2*, *4* and *6* respectively, for various values of $\alpha_2 T_0$. For any given value of L, the instability threshold is given by the minimum of M with respect to a for various values of $\alpha_2 T_0$. We observe from Figs 1a-d that the critical Marangoni number does significantly depend upon whether the liquid layer is relatively hotter or cooler, and the hotter the liquid layer the more the onset of instability is postponed provided α_2 is positive, and for the liquid layer for which α_2 is negative the onset of instability would be preponed for the hotter layer. In other words, for a fixed value of L, the onset of instability is preponed or postponed for a cooler layer than for a relatively hotter layer of the same liquid for $\alpha_2 > 0$ or $\alpha_2 < 0$.

The wave number a_c remains unchanged for various values of $\alpha_2 T_0$ as shown in Fig.2.

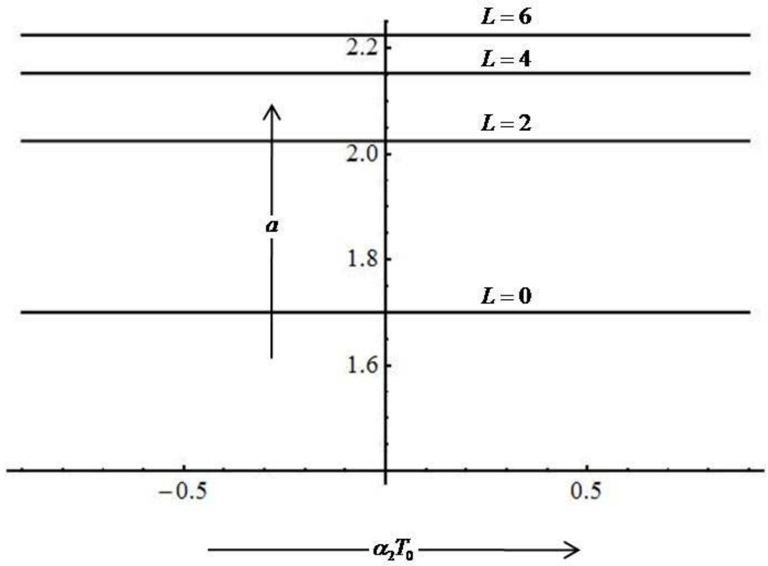

Fig.2. Wave number a_c as a function of $\alpha_2 T_0$ when L = *0*, *2*, *4* and *6*.

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Nomenclature

$\sqrt{a_x^2 + a_y^2}$ – resultant wave number
d – depth of the liquid layer in the unperturbed state
k – thermal conductivity
q – heat transfer coefficient between the upper free surface and the air phase
T_0 – temperature of the lower boundary surface
T_1 – temperature of the upper boundary surface
w – z-component of velocity perturbation
α_2 – the coefficient of variation of specific heat at constant volume
θ – z-component of temperature perturbation
κ – thermal diffusivity
ν – kinematic viscosity
ρ – density
σ – rate of change of surface tension with temperature

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